Math Night IV: Algorithms

Principia

Problems are roughly ordered by difficulty ($\star$ to $\star\star\star$). Hints are collected at the end.

Problem 1 (The Pecking Order). $\star$

In a flock of n chickens, every pair has fought exactly once, and there are no draws.

(a) Show that the chickens can be lined up so that each chicken defeated the one immediately behind her.

(b) Show that some chicken is a *queen*: for every other chicken x, she either defeated x, or defeated some chicken who defeated x.

Problem 2 (The Laser in the Box). $\star$

A rectangular mirrored box is m units wide and n units high, where m and n are positive integers. A laser is fired from the lower left hand corner at $45°$, heading upward and to the right.

(a) Show that the beam eventually reaches another corner - it can't bounce forever.

(b) Which corner does it reach?

(c) How many non-corner wall bounces occur along the way?

(d) You've just built an optical computer. Where does it compute $\gcd(m, n)$?

Problem 3 (The Eroding Skyline). $\star\star$

A row of n mountains has distinct heights. The two mountains at the ends can never disappear.

At each step, you may erase any other mountain that is shorter than its nearest remaining mountain on both sides. The gap then closes.

(a) Show that the final skyline does not depend on the order in which mountains are erased.

(b) Describe the surviving mountains directly from the original skyline, without running the process.

(c) If the heights are placed in uniformly random order, what is the expected number of survivors?

Problem 4 (The Royal Mint). $\star\star$

A kingdom mints coins in three positive integer denominations 1, a, and b, with $1 < a < b$. Every merchant makes change by repeatedly handing over the largest coin that fits.

(a) The royal treasurer claims that this always uses the fewest possible coins. Prove him wrong.

(b) Write

$$b = qa + r, \qquad 0 \le r < a.$$

Determine exactly for which currencies $\{1, a, b\}$ the greedy procedure is optimal for *every* amount.

Problem 5 (The Safecracker). ⋆⋆

A vault keypad has digits 0–9 and no enter key. It silently watches the last four digits typed and unlocks as soon as they match its four digit code. Leading zeroes are allowed. You have forgotten the code.

Trying all 10,000 codes separately can take 40,000 keystrokes. Prove that a clever safecracker needs *exactly*

$$10{,}003$$

keystrokes in the worst case.

Problem 6 (The Gladiators). ⋆⋆

Two stables of gladiators meet in the arena. Every gladiator has a positive real strength. When gladiators of strengths x and y fight, the first wins with probability

$$\frac{x}{x+y},$$

and the winner absorbs the loser's strength. Each fight pits one surviving gladiator from each stable against the other. Before every fight, each manager may choose whom to send, using the entire history so far. Fights continue until one stable is empty.

Show that the probability a given stable wins is independent of every choice made by both managers, and compute that probability from the initial strengths.

Problem 7 (The Compromise Committee). ⋆⋆

There are $n \geq 3$ delegates, initially supporting n distinct proposals

$$x_1 < x_2 < \cdots < x_n.$$

At each meeting, the chair chooses any three delegates. All three then adopt the median of their three current proposals.

(a) Which of the original proposals can the chair force everyone eventually to support?

(b) If the target is x_k, what is the fewest number of meetings required?

(c) If nobody controls the meetings, must the delegates eventually reach agreement?

Problem 8 (The Averaging Conference). ⋆⋆

Each of n scientists begins with a real number. In one round, the scientists are divided into disjoint pairs, with at most one scientist sitting out. In every pair, both scientists replace their numbers by their arithmetic mean. A scientist sitting out keeps her number.

For which values of n is there a finite schedule of pairings, depending only on n, that leaves every scientist holding the exact original average no matter what numbers they started with?

When it is possible, what is the fewest number of rounds required?

Problem 9 (The One Way Street). ⋆⋆

A one way street has n parking spots. One by one, n drivers arrive. Driver i has a preferred spot $p(i)$. Each driver goes directly to that spot. If it is occupied, the driver continues forward and takes the first empty spot. A driver who reaches the end without parking leaves, very unhappy.

(a) For $n = 2$, check that exactly three of the four preference lists allow everyone to park.

(b) How many preference lists exist wherein everyone parks?

Problem 10 (The Bureaucrats). ⋆⋆

An office corridor has n desks in a row and begins with finitely many files. Whenever a desk holds at least two files, that bureaucrat may *process*: one file is sent to the left and one to the right. A file sent past either end of the corridor leaves the building forever.

At each step, an office manager chooses any overloaded desk to process. The procedure stops when every desk contains at most one file.

(a) Show that the workday always ends.

(b) Show that the final arrangement of files is independent of every choice the manager made.

(c) ⋆⋆⋆ Show that even the number of times each individual desk processes is independent of the choices.

Problem 11 (The Library Mob). ⋆⋆

The library's n books have distinct call numbers and have been shelved in a scrambled order. At each step, a librarian chooses any two books that are out of order (the book farther left has the larger call number) and swaps them. Nobody coordinates. This continues until no such pair remains.

(a) Show that the mob always finishes.

(b) Two rival mobs begin with identical scrambled shelves. Show that their total numbers of swaps agree modulo 2, regardless of what anyone did.

(c) In terms of the starting arrangement, determine the largest and smallest possible numbers of swaps.

(d) ⋆⋆⋆ Show that every swap count permitted by parts (b) and (c) actually occurs.

Problem 12 (The Hall of Twins). ⋆⋆

In a court of n courtiers, each is secretly Loyal or Treacherous. Your only instrument is a private chamber: send in any two courtiers, and you learn whether they are of the same kind or of different kinds. You don't learn anything else.

Your task is to produce one courtier of the majority kind, or to declare a tie when n is even and the two kinds are equally numerous.

(a) The obvious method uses $n-1$ chamber visits. When $n=5$, succeed in only three.

(b) Give a strategy using

$$n-\nu(n)$$

chamber visits, where $\nu(n)$ is the number of 1s in the binary representation of n.

(c) **Advanced challenge, very hard.** Show that no strategy can do better. This lower bound is substantially trickier than every other problem on the sheet.

Hints

Read only as needed. For the harder problems, the hints become progressively stronger.

1. The Pecking Order. For (a), suppose $n-1$ chickens are already in a line. Where can the last chicken be inserted? For (b), consider a chicken with the largest possible number of wins.

2. The Laser in the Box. Instead of reflecting the beam, reflect the box. The bouncing path becomes one straight line through a grid of mirrored copies.

3. The Eroding Skyline. A mountain that is taller than every mountain to its left can never be erased. The same is true from the right. Then prove no other mountain can remain. For the expectation, count records.

4. The Royal Mint. Watch what greed does at the first multiple of a strictly beyond b. Then prove that this is the only possible obstruction.

5. The Safecracker. Use a directed graph whose vertices are strings of three digits and whose edges are strings of four digits. What kind of walk uses every edge exactly once?

6. The Gladiators. Track the total strength currently belonging to one stable. Compute its expected change in a single fight, then ask what values it can have when the contest ends.

7. The Compromise Committee. Relative to a target proposal x_k, track how many delegates are currently below it and how many are currently above it. To see that uncontrolled meetings may fail to converge, try five delegates and look for a two-state cycle.

8. The Averaging Conference. First try the initial readings $1, 0, 0, \ldots, 0$. What denominators can repeated pairwise averaging create? For the construction when n is a power of two, label the scientists by binary strings.

9. The One Way Street. Add an $(n+1)$st spot and bend the street into a circle. Every preference list now parks all n drivers and leaves one spot empty. Use rotational symmetry carefully.

10. The Bureaucrats. First, assign desk i the weight $i(n+1-i)$ and track the total weighted number of files. Next, compare two legal moves performed in opposite orders. For part (c), compare the two vectors of processing counts and study their difference.

11. The Library Mob. First count inversions. For the minimum, also view the shelf order as a permutation written in cycles. For part (d), find a legal swap that removes at least three inversions and replace that one swap by three legal swaps with the same final effect.

12. The Hall of Twins. For (a) and (b), maintain groups known to be internally identical, of sizes $1, 2, 4, 8, \ldots$. Equal-sized groups that test different cancel; equal-sized groups that test the same merge. This is binary carrying. Part (c) needs a separate adversary lower-bound argument; the construction alone does not contain its proof.