Math Night III

Principia

May 9, 2026

Problems are roughly ordered by difficulty ($\star$ to $\star\star\star$).

Problem 1 (Sharing Your Birthday). $\star$

In a room of 23 people, there is a better-than-even chance that two share a birthday. This is the famous birthday paradox. Ignore leap years.

Now suppose your birthday is May 9. How many other people *need to be in the room before there is a better-than-even chance that someone shares your specific birthday?*

HINT. The probability that one particular person does *not* share your birthday is 364/365.

Problem 2 (String Around the Earth). $\star$

A string is wrapped snug around the Earth's equator. Treat the Earth as a perfect sphere: no mountains, no oceans, no mess.

You want to lift the string off the ground by exactly 1 meter at every point.

How much additional string do you need?

HINT. Compare the circumferences of circles of radius R and $R + 1$.

Problem 3 (The Long Bus to Math Night). $\star$

The bus to Math Night alternates between arriving after 1 minute and arriving after 19 minutes. So the average gap between buses is 10 minutes.

You arrive at a uniformly random time during a long stretch of this schedule.

What is your expected wait?

HINT. You are much more likely to arrive during a long gap than a short gap.

Problem 4 (The Random Class Size). $\star$

A school has one seminar with 5 students and one lecture with 95 students.

A uniformly random class has average size

$$\frac{5 + 95}{2} = 50.$$

A uniformly random student is asked how large her class is. What is the expected answer?

Why are these two averages different?

HINT. A student is not equally likely to be in each class.

Problem 5 (The Dog and the Lamppost). ⋆

A dog is tied to a lamppost by a long leash. It runs around the lamppost n times. Each time, it independently chooses clockwise or counterclockwise with equal probability.

At the end, you pull the leash tight.

(a) *What is the probability the dog is still wrapped around the pole?*

(b) *What is the expected absolute number of wraps around the pole?*

HINT. Clockwise is $+1$, counterclockwise is -1. The only thing that survives pulling tight is the net winding number.

Problem 6 (The Necklace Cut). ⋆

A circular necklace has n beads, each independently colored red or blue with equal probability. You cut it at one of the n gaps, chosen uniformly at random, and lay it out as a string.

A color-run *is a maximal consecutive block of beads of the same color. For example, $RRBBBRR$ has three color-runs.*

How many color-runs do you expect to see in the resulting string?

HINT. A run begins at the first bead, and then again whenever the color changes from one bead to the next.

Problem 7 (The Coin Factory). ⋆

A factory makes coins with random biases. For each coin, its probability of heads p is chosen uniformly from 0 to 1. You pick a coin and flip it three times. It comes up heads all three times.

What is the probability the next flip is heads?

HINT. After observing the flips, not all values of p are equally plausible anymore.

Problem 8 (Nontransitive Dice). ⋆

Here are three strange dice:

$$A = (2, 2, 4, 4, 9, 9), \qquad B = (1, 1, 6, 6, 8, 8), \qquad C = (3, 3, 5, 5, 7, 7).$$

Show that

$$A \text{ usually beats } B, \qquad B \text{ usually beats } C, \qquad C \text{ usually beats } A.$$

What is $\Pr(A > B)$*? What is* $\Pr(B > C)$*? What is* $\Pr(C > A)$*?*

HINT. There are 36 equally likely pairs of rolls for each comparison.

Problem 9 (The Coupon Collector). ⋆

A cereal company puts a small toy in every box. There are n different toys, each equally likely to appear, independently across boxes. You want all of them.

What is the expected number of boxes you must buy?

HINT. After you have collected $k-1$ distinct toys, what is the probability the next box contains a new one?

Problem 10 (The Bag of Loose Ends). ⋆⋆

You put n shoelaces in a bag. Their $2n$ ends stick out. You repeatedly choose two loose ends uniformly at random and tie them together, until every end is tied.

At the end, the shoelaces have become some number of closed loops.

What is the expected number of loops?

HINT. When there are k open chains left, what is the probability that the next knot ties together the two ends of the same chain?

Problem 11 (One Giant Necklace). ⋆⋆

You again put n shoelaces in a bag and randomly tie pairs of loose ends until every end is tied.

What is the probability that the result is one single giant loop?

HINT. To get one giant loop, you must avoid closing a smaller loop until the very last tie.

Problem 12 (Penney's Coins). ⋆⋆

You and a friend each pick a sequence of three coin flips: you pick HHH, your friend picks HTH. You then flip a fair coin repeatedly and track the running sequence. Whoever's pattern appears first wins.

Both patterns appear with probability $1/8$ in any given window of three flips. So the expected number of flips until you see HHH should equal the expected number until you see HTH.

It doesn't.

What is the expected wait for each pattern? And which pattern wins more often?

HINT. Keep track of how much of your target pattern you have already matched. The pattern HHH overlaps with itself differently from HTH.

Problem 13 (The Three Apprentices). ⋆⋆

A wizard summons three apprentices and announces:

"I am about to place a hat on each of your heads. Each hat is independently red or blue, with equal probability. You will see the others' hats but not your own. At my signal, you must each—simultaneously—either guess the color of your own hat or pass. If at least one of you guesses correctly and none guess incorrectly, you all pass. Otherwise, you all fail."

"You may agree on a strategy now. You may not communicate after the hats are on."

What is the best strategy? What probability of passing does it achieve?

HINT. It is better to lose on a few configurations with certainty than to risk losing on many configurations with small probability.

Problem 14 (The Bad Host). ⋆⋆

There are three doors. One has a car; two have goats. You pick Door 1.

The host opens a goat door. But the host is not Monty Hall. If he has a choice, he opens Door 2 with probability q and Door 3 with probability $1-q$, where $0 \le q \le 1$.

He opens Door 2.

Should you switch? How does the answer depend on q?

Hint. Compute Pr(car behind Door 3 | host opened Door 2).

Problem 15 (Bertrand's Chord). ⋆⋆

What is the probability that a "random" chord of a unit circle is longer than the side of an equilateral triangle inscribed in that circle?

You will find that three perfectly natural ways of defining "a random chord" give three different answers:

(a) *pick a uniformly random pair of endpoints on the circle;*

(b) *pick a uniformly random midpoint inside the disk;*

(c) *pick a uniformly random distance from the center along a random radius.*

Compute all three. Why doesn't "random" pick itself?

Hint. A chord is long enough exactly when its midpoint lies within radius 1/2 of the center.

Problem 16 (The Ant on a Cube). ⋆⋆

An ant sits at one corner of a cube. Each second, it walks along one of the three edges meeting its current vertex, choosing uniformly at random.

What is the expected number of seconds until it first reaches the diagonally opposite corner?

Hint. The ant's exact vertex matters less than its graph distance from the target corner.

Problem 17 (The Ant on a Rubber Band). ⋆⋆

A rubber band is stretched between two points, 1 km apart. An ant sits at one end. The ant begins walking toward the other end at 1 cm per second. At the end of each second, the rubber band stretches uniformly by an additional 1 km—so after t seconds, the band is $t+1$ km long.

Does the ant ever reach the far end?

Hint. Track the ant's fractional position along the band, not its absolute distance.

Problem 18 (The Random Hair Braid). ⋆⋆

Three colored strings hang vertically: red, blue, green. At each step, you randomly choose either the left pair or the right pair, with equal probability, and cross those two strings. Ignore over/under information; only track which string occupies each position.

After n crossings, where is the red string?

More precisely: if the red string starts on the left, what is the probability it is on the left after n random crossings?

Hint. Track only the red string's position: left, middle, or right. This is a three-state Markov chain.

Problem 19 (The Two Envelopes). ⋆⋆

A stranger hands you a sealed envelope. "There's money inside," he says. "I'll tell you only that the other envelope contains either twice as much, or half as much. You may keep this one, or trade for the other."

You open yours and find X dollars. Expected value of switching:

$$\frac{1}{2}(2X) + \frac{1}{2}(X/2) = 1.25X.$$

So you should switch—gain in expectation. But by symmetry, the same calculation applies whichever envelope you started with. So you should always switch—even before opening either one.

Where is the error?

Hint. After seeing X, is it really probability $1/2$ that the other envelope contains $2X$ and probability $1/2$ that it contains $X/2$?

Problem 20 (The 100 Prisoners). ⋆⋆⋆

The warden tells 100 prisoners, each numbered 1 through 100:

"Tomorrow, in a separate room, I will place 100 numbered boxes in a row. Each box will contain one slip of paper bearing one of your numbers, with the assignment of slips to boxes chosen uniformly at random. One by one, you will enter the room. Each of you may open up to 50 boxes, in any order, but you must leave the room exactly as you found it. You may not communicate after the procedure begins. If every single prisoner finds the slip with his own number, you all go free. If even one of you fails, you all hang."

Without a strategy, the probability of survival is essentially zero—about 10^{-30}. *The prisoners gather to plan.*

Find a strategy that gives them better than 30% chance of survival.

Hint. Think of the slips as defining a permutation. What if each prisoner follows a cycle?

Problem 21 (The Pirate Council). ⋆⋆⋆

100 pirates, ranked strictly from 1 (most junior) to 100 (most senior), have captured 100 gold coins. Their custom is as follows: the most senior surviving pirate proposes a distribution of the coins. All pirates—including the proposer—vote yes or no. If at least half vote yes, the proposal passes and the coins are distributed accordingly. Otherwise, the proposer is thrown overboard, and the next-most-senior pirate proposes.

The pirates are perfectly rational and perfectly informed. Each prioritizes, in this order: (1) staying alive, (2) maximizing her own gold, (3) seeing other pirates die.

What does pirate 100 propose? How does she keep her head and as much gold as possible?

Hint. Start with 1 pirate, then 2, then 3. The convention that ties pass matters.

Problem 22 (The Drunkard's Walk). ⋆⋆⋆

A drunk takes random unit steps on the integer lattice in d dimensions: at each step, he picks one of the $2d$ directions uniformly at random.

For $d = 1$ and $d = 2$, he eventually returns to his starting point with probability 1. For $d = 3$ and higher, there is positive probability that he never returns.

Assume that the probability of being back at the origin after $2n$ steps is on the order of $1/n^{d/2}$. Use this to explain where the difference comes from.

Kakutani: "A drunk man will find his way home, but a drunk bird may be lost forever."

HINT. Sum the probabilities of being back at the origin over all times. Which dimensions give a divergent series?

Problem 23 (The Two-Post Leash). ⋆⋆⋆

A dog is tied to a wall by a leash. There are two posts in the yard, called A and B. The dog walks around the posts, sometimes clockwise and sometimes counterclockwise, and eventually returns to where it started. Then you pull the leash tight.

Model the dog's route by a word in the letters A, B, A^{-1}, B^{-1}. A clockwise loop around post A is A, a counterclockwise loop around post A is A^{-1}, and similarly for B. Model "pulling tight" by repeatedly canceling adjacent inverse pairs such as AA^{-1} or $B^{-1}B$. The leash pulls free if the word reduces to the empty word.

With one post, the only thing that matters is the net winding number: clockwise counts as $+1$, counterclockwise as -1.

With two posts, can a leash have net winding number zero around each post and still fail to pull free?

Problem 24 (The Invisible Spy). ⋆⋆⋆

An invisible spy travels the plane at speed 1, along any path she likes. You are never told where she is.

You build fence. You may build at finitely many points at once, in any directions, at total rate r—so, for example, you could grow circular arcs at three different points with rates summing to r. The spy cannot cross fence, but she cannot see it either.

(a) *What is the infimum of r for which some strategy guarantees that the spy cannot escape to infinity?*

(b) *What is the infimum of r if you only require capture with probability 1?*