Math Night II

Principia

March 14, 2026

Problems are roughly ordered by difficulty ($\star$ to $\star\star\star$).

Problem 1 (The Broken Stick). $\star$

A stick of unit length is broken at two points chosen independently and uniformly at random. Three pieces result.

(a) What is the probability the three pieces form a triangle?

(b) What is the probability the triangle is acute*?*

Explain where π appears in part (b) *and why.*

Hint. Write the lengths as $(x, y, 1-x-y)$ in the unit simplex. For (a) the triangle inequality gives a clean region. For (b) the Pythagorean constraint introduces a circular boundary.

Problem 2 (The Random GCD). $\star$

Pick two large positive integers at random.

(a) What is the probability they share no common factor? That is, compute

$$\lim_{N\to\infty} \frac{\#\big\{(a,b) \in \{1,\dots,N\}^2 : \gcd(a,b)=1\big\}}{N^2}.$$

(b) Explain why π appears in the answer to a question about divisibility that mentions no circles, no angles, and no geometry.

Hint. For each prime p, the probability p divides both numbers is $1/p^2$. The primes act independently. You will need Euler's product formula for $\zeta(2)$.

Problem 3 (Geometry Without Circles). $\star$

For $p \geq 1$, define the L^p norm $\|(x,y)\|_p = \big(|x|^p + |y|^p\big)^{1/p}$ and the unit ball $B_p = \{(x,y) : \|(x,y)\|_p \leq 1\}$.

(a) Draw B_1 and B_∞ and compute their areas.

(b) Show that

$$\mathrm{Area}(B_p) \;=\; \frac{4\,\Gamma\big(1+\frac{1}{p}\big)^2}{\Gamma\big(1+\frac{2}{p}\big)}.$$

(c) Check that when $p=2$ the area equals π.

Problem 4 (The Laser Field). $\star\star$

A spy must cross a long corridor. The floor is ruled with parallel laser beams spaced d apart. She is carrying a rigid rod of length $\ell < d$. She trips and drops it; the rod lands at a uniformly random position and angle.

(a) *What is the probability the rod crosses a beam?*

(b) *The spy reasons: "If I bend my rod into a circle, maybe it will cross fewer beams." She bends it into a closed curve of total arc length ℓ—a circle, a zigzag, the letter S, anything she likes. Does the expected number of crossings depend on the shape?*

Hint. For (a), integrate over the angle the rod makes with the beams. For (b), prove the result for a line segment first, then use linearity of expectation.

Problem 5 (The Drunkard, the Dog, and the Bird). ⋆⋆

A drunkard stands at the origin of $\mathbb{Z}^d$. Each second he steps to a uniformly random neighboring lattice point.

(a) ***The hallway*** *($d = 1$). Show that he returns to the origin with probability 1.*

(b) ***The city grid*** *($d = 2$). Show the same.*

(c) ***Open air*** *($d = 3$). Show the return probability is strictly less than 1.*

Shizuo Kakutani summarized: A drunk man will find his way home, but a drunk bird may get lost forever.

Where does π enter? In which step of which proof?

Hint. The probability of being at the origin after $2n$ steps in $d = 1$ is $\binom{2n}{n}/4^n \sim 1/\sqrt{\pi n}$ by Stirling. Recurrence follows from the divergence of $\sum 1/\sqrt{n}$.

Problem 6 (Lattice Visibility). ⋆⋆

*You stand at the origin in $\mathbb{Z}^2$. An opaque tree of zero width is planted at every lattice point $(m, n) \neq (0, 0)$. A tree at (m, n) is **visible** if $\gcd(|m|, |n|) = 1$.*

Compute the natural density of visible lattice points:

$$\lim_{R\to\infty} \frac{\#\{(m,n) \in \mathbb{Z}^2 \setminus \{0\} : m^2 + n^2 \le R^2,\ \gcd(|m|, |n|) = 1\}}{\#\{(m,n) \in \mathbb{Z}^2 \setminus \{0\} : m^2 + n^2 \le R^2\}}.$$

Explain why π is hiding in your answer.

Problem 7 (The Coin-Flip Oracle). ⋆⋆

A fortune teller claims she can compute π with nothing but a fair coin. Her method: flip a coin repeatedly until the cumulative number of heads first exceeds the cumulative number of tails. Record the fraction heads/total flips. Start over and repeat many times. Average the recorded fractions.

She claims the average converges to $\pi/4$. Is she right?

Hint. Model as a symmetric random walk $S_0 = 0$, $S_n = S_{n-1} \pm 1$. Let τ be the first hitting time of $+1$. The stopping time $\tau = 2k + 1$ occurs with probability $C_k/4^{k+1}$, where $C_k = \binom{2k}{k}/(k + 1)$ is the k-th Catalan number.

Problem 8 (Circular Coins). ⋆⋆⋆

Coins of weights $1, 2, \ldots, n$ are placed around a circle. The weights increase consecutively, but the starting point and direction (clockwise or counterclockwise) are unknown. All coins look identical.

You may compare any two coins using a balance scale (one coin per side).

Baron Munchausen claims that using at most k weighings you can always determine the exact weight of at least one coin.

Determine the largest n for which this is possible.